2022

Time - 3 hours

Full Marks - 80

*Answer **all groups** as per instructions.*
Figures in the right hand margin indicate marks.
The symbols used have their usual meaning.

GROUP - A

1. Answer all questions by choosing correct option. [1 × 12

(a) Which of the following is true ?

(i) Finite union of open sets is open

(ii) Arbitrary union of open sets is open

(iii) Both (i) and (ii) (iv) None of the above

(b) The $S=\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\} \cup \{0\}$ is

(i) Open (ii) Closed

(iii) Bounded (iv) Both (ii) and (iii)

(c) Let (x_n) be a Cauchy sequence. Then the sequence is ....

(i) Bounded (ii) Increasing

(iii) Decreasing (iv) All of the above

(d) Let (x_n) be a cauchy sequence. If the sequence is bounded, then

(i) $\lim x_n = \sup x_n$ (ii) $\lim x_n = \inf x_n$

(iii) Both (i) and (ii) (iv) None of the above

(e) Let f : X → **R** be continuous on X. Which of the following is true ?

(i) If O is an open subset of X, then f(O) is an open subset in Y.

(ii) If O is an open subset of Y, then $f^{-1}(O)$ is an open subset in X.

(iii) If F is a closed subset of X, then f(F) is a closed subset in Y.

(iv) All of the above

(f) Let f : **R** → **R** be a function defined by

$$f(x)=\begin{cases}1 , & x \text{ is rational} \\ 0 , & x \text{ is irrational.}\end{cases}$$

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Then which of the following is true ?

(i) f is continuous only at 0

(ii) f is continuous at every rationals

(iii) f is continuous at every irrationals

(iv) f is nowhere continuous.

(g) Let $f(x) = x \sin \frac{1}{x}$, $(x \neq 0)$ and $f(0) = 0$. Which of the following is true ?

(i) f is continuous at 0

(ii) f is continuous everywhere

(iii) f is not differentiable at 0

(iv) All of the above

(h) State Caratheodory's Theorem.

(i) Give an example of a bounded sequence which is not Cauchy.

(j) Give an example of a set which is bounded above but unbounded below

(k) Give an example of a sequence which is convergent but neither monotonically increasing nor decreasing.

(l) Find the interior points of the set $(0, 1) \cup \{2, 3\}$.

GROUP - B

2. Answer <u>any eight</u> questions. [2 × 8

(a) State Weierstress-Completeness Principle.

(b) Define the term limit point of a set.

(c) Is the set $F = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\ldots\right\}$ a closed set in **R** ? Justify.

(d) State Bolzano-Weierstrass Theorem for sequences.

(e) Show that $x_n = (1/n)_{n \in N}$ is a Cauchy sequence.

(f) Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

(g) State the location of roots theorem.

(h) If $f : X \to \mathbf{R}$ is continuous on X, is necessarily f uniformly continuous on X ? Justify your answer.

(i) State Mean-Value Theorem.

(j) Test the differentiability of $f(x) = x\,|\,x\,|^2$ at origin.

GROUP - C

3. Answer <u>any eight</u> questions. [3 × 8

(a) Is every bounded above subset of an ordered field F has least upper bound ? Justify your answer.

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(b) Show that every finite subset of R is a closed set.

(c) If $0 \le | a - b | < \varepsilon$, for every ε. Then show that $a = b$.

(d) Is every Cauchy sequence bounded ? Justify your answer.

(e) If $x_n \to l$ as $n \to \infty$, then show that $|x_n| \to |l|$.

(f) Let $f : X \to R$ is differentiable at a point $a \in X$. Is necessarily f continuous at a ? Justify your answer. What about the converse ?

(g) Test the continuity of the function $\sin \frac{1}{x}$ at 0.

(h) Is necessarily the function $f : (0, 1) \to R$ defined by $f(x) = \frac{1}{x}$ uniformly continuous ? Justify.

(i) Show that $\frac{x}{1+x} < \log(1+x) < x, \forall x > 0$.

(j) Let f be differentiable on (a, b). If $f'(x) \ge 0, \forall x \in (a,b)$. Then show that f is monotonically increasing.

GROUP - D

4. Answer any four questions. [7 × 4

(a) State and prove Archimedean Principle. Use Archimedean Principle to show that inf $\{x \in R : x < 0\}$. [5 + 2

(b) Prove that a subset of **R** is closed if and only if contains all its limit points.

(c) Show that a bounded sequence has a convergent subsequence.

(d) State and prove Leibnitz Alternating Series Test.

(e) Show that the image of a closed bounded set under a continuous function is closed and bounded.

(f) Let S be a closed and bounded subset of **R**. If $f : S \rightarrow \mathbf{R}$ is continuous on S, then prove that f is uniformly continuous on S.

(g) State and prove Rolle's Theorem. Use Rolle's Theorem to show that the equation $10x^4 - 6x + 1 = 0$ has a root between 0 and 1.

[5 + 2